

A note on Berezin-Toeplitz quantization of the Laplace operator

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Abstract

Given a Hodge manifold, it is introduced a self-adjoint operator on the space of endomorphisms of the global holomorphic sections of the polarization line bundle. Such operator is shown to approximate the Laplace operator on functions when composed with Berezin-Toeplitz quantization map and its adjoint up to an error which tends to zero when taking higher powers of the polarization line bundle.

1 Introduction

Let M be a n -dimensional projective manifold and let g be a Hodge metric on M . This means that M is equipped with a complex structure J and with a positive Hermitian line bundle (L, h) . Denoted by Θ the curvature of the Chern connection, the form $\omega = 2\pi i\Theta$ is positive, and it holds $g(u, v) = \omega(u, Jv)$. Let

$$\Delta : C^\infty(M) \rightarrow C^\infty(M)$$

be the positive Laplacian associated with the metric g (recall that it is defined by $\Delta(f)\omega^n = -n i\partial\bar{\partial}f \wedge \omega^{n-1}$ for any complex-valued smooth function f on M). In this note it will be shown that Δ is approximated in a suitable sense by a sequence of self-adjoint positive operators

$$\Delta_m : V_m \rightarrow V_m$$

acting on finite dimensional Hermitian vector spaces V_m (see definitions at Sections 2 and 4). To be a little more precise, it will be proved that there exist maps

$$T_m : C^\infty(M) \rightarrow V_m, \quad T_m^* : V_m \rightarrow C^\infty(M),$$

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in fact adjoint to each other with respect to suitable Hermitian products, such that

$$T_m^* \circ \Delta_m \circ T_m(f) = m^{n-1} \Delta f + O(m^{n-2}) \quad (1)$$

as $m \rightarrow \infty$ for any given smooth function f . For any $m > 0$ the map T_m is the well known Berezin-Toeplitz quantization map, and the operator Δ_m depends only on the projective geometry of the Kodaira embedding of M via L^m . Moreover Δ_m is related to the metric g via the Fubini-Study metric induced by the L^2 -inner product on the space of global holomorphic sections of L^m (see Section 4). Thanks to results available on asymptotic expansions of Bergman kernel [5] and Toeplitz operators [6], what one can prove is indeed the following result, which obviously implies (1).

Theorem 1.1. *There is a complete asymptotic expansion*

$$T_m^* \circ \Delta_m \circ T_m(f) = \sum_{r \geq 0} P_r(f) m^{n-1-r} + O(m^{-\infty}),$$

where P_r are self-adjoint differential operators on $C^\infty(M)$. More precisely, for any $k, R \geq 0$ there exist constants $C_{k,R,f}$ such that

$$\left\| T_m^* \circ \Delta_m \circ T_m(f) - \sum_{r=0}^R P_r(f) m^{n-1-r} \right\|_{C^k(M)} \leq C_{k,R,f} m^{n-R-2}.$$

Moreover one has

$$P_0(f) = \Delta f, \quad P_1(f) = -\frac{1}{2\pi} \Delta^2 f.$$

The construction of the quantized Laplacian Δ_m was inspired by a work of J. Fine on the Hessian of the Mabuchi energy [2]. Even though in principle Δ_m is unrelated to the problem of finding canonical metrics on M , when ω is balanced in the sense of Donaldson (see definition recalled at Section 6) the relation between Δ_m and Δ is even more evident as shown by the following

Theorem 1.2. *If ω is m -balanced then*

$$\Delta_m(A) = C T_m \circ \Delta \circ T_m^*(A)$$

for all $A \in V_m$, where $C = \frac{m^{n-1} \left(\int_M \frac{\omega^n}{n!} \right)^2}{(\dim H^0(M, L^m))^2}$.

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2 The space V_m

The space V_m is nothing but $\text{End}(H_m)$, being $H_m = H^0(M, L^m)$ the space of the holomorphic sections of L^m . By Riemann-Roch theorem $\dim V_m$ grows like a positive multiple of m^{2n} when $m \rightarrow \infty$. The space H_m is equipped with a Hermitian inner product b_m induced by the Hermitian metric h^m on L^m and the Kähler form ω . Explicitly it is given by

$$b_m(s, t) = \int_M h^m(s, t) \frac{\omega^n}{n!} \quad (2)$$

for all $s, t \in H_m$. Thus V_m is a Hermitian vector space with inner product defined by

$$\langle A, B \rangle = \text{tr}(AB^*), \quad (3)$$

for all $A, B \in V_m$. Here B^* denotes the adjoint of B with respect to b_m .

3 The maps T_m and T_m^*

The map $T_m : C^\infty(M) \rightarrow V_m$ is the well known Berezin-Toeplitz quantization operator [5]. Given a smooth function f on M , the operator $T_m(f)$ is the composition $T_m(f) = P \circ M(f)$, where $M(f) : H_m \rightarrow \Gamma(M, A^m)$ is the multiplication by f :

$$M(f)(s) = fs,$$

and $P : \Gamma(M, A^m) \rightarrow H_m$ is the orthogonal projection with respect to the obvious extension of the inner product b_m to smooth sections.

The space of smooth function $C^\infty(M)$ is equipped with the L^2 -product induced by ω , given by

$$\langle f, g \rangle = \int_M f \bar{g} \frac{\omega^n}{n!}, \quad (4)$$

for all $f, g \in C^\infty(M)$. Let $T_m^* : V_m \rightarrow C^\infty(M)$ be the adjoint of T_m .

Lemma 3.1. *Let $\{s_\alpha\}$ be an orthonormal basis of H_m . For all $A \in V_m$ it holds:*

$$T_m^*(A) = \sum_{\alpha} h^m(A s_\alpha, s_\alpha).$$

Proof. It is an easy consequence of general theory. For every $f \in C^\infty(M)$ one has

$$\text{tr}(A T_m(f)^*) = \sum_{\alpha} b_m(A s_\alpha, T_m(f) s_\alpha).$$

Substituting

$$T_m(f) s_\alpha = \sum_{\beta} \left(\int_M f h^m(s_\alpha, s_\beta) \frac{\omega^n}{n!} \right) s_\beta,$$

it follows

$$\mathrm{tr}(A T_m(f)^*) = \sum_{\alpha} \int_M \bar{f}(x) h^m(A s_{\alpha}, s_{\alpha}) \frac{\omega^n}{n!},$$

which gives the thesis by arbitrariness of f after noting that

$$\int_M T_m^*(A) \bar{f} \frac{\omega^n}{n!} = \mathrm{tr}(A T_m(f)^*).$$

□

Note that the map T_m^* takes an endomorphisms $A \in V_m$ to the restriction to the diagonal of its integral kernel. More precisely, given an orthonormal basis $\{s_{\alpha}\}$ of H_m , the integral kernel of A is the smooth section $K(A)$ of $L^m \boxtimes L^{-m}$ over $M \times M$ given by

$$K(A)(x, y) = \sum_{\alpha, \beta} \int_M h^m(A s_{\alpha}, s_{\beta})(z) s_{\beta}(y) \otimes s_{\alpha}^*(x) \frac{\omega_z^n}{n!},$$

where $s_{\alpha}^*(x)$ is the metric dual of $s_{\alpha}(x)$ in the fiber of L^m over the point x . The restriction of the kernel to the diagonal is (naturally identified with) the smooth function $T_m^*(A)$ thanks to Lemma 3.1. When A is of the form $T_m(f)$ for some smooth function f , the integral kernel is given by

$$K(T_m(f))(x, y) = \sum_{\alpha, \beta} \int_M f(z) h^m(s_{\alpha}, s_{\beta})(z) s_{\beta}(y) \otimes s_{\alpha}^*(x) \frac{\omega_z^n}{n!},$$

whence

$$T_m^* \circ T_m(f)(x) = \sum_{\alpha, \beta} \int_M f(z) h^m(s_{\alpha}, s_{\beta})(z) h^m(s_{\beta}, s_{\alpha})(x) \frac{\omega_z^n}{n!}.$$

For a constant function $f = c \in \mathbf{R}$, one has

$$T_m^* \circ T_m(c) = c \rho_m,$$

where $\rho_m = \sum_{\alpha} |s_{\alpha}|_{h^m}^2$ is the so-called Bergman kernel of ω .

4 The operator Δ_m

The operator $\Delta_m : V_m \rightarrow V_m$ is a self-adjoint operator which depends just on projective geometry of M in $\mathbf{P}(H_m)$. Consider the embedding

$$\iota_m : M \rightarrow \mathbf{P}(H_m),$$

given by the Kodaira map of M in $\mathbf{P}(H_m^*)$ induced by L^m , followed by the isomorphism $\mathbf{P}(H_m^*) \simeq \mathbf{P}(H_m)$ induced by the Hermitian product b_m .

Every endomorphism A of H_m induces a (holomorphic) vector field $\nu(A)$ on $\mathbf{P}(H_m)$ whose flow is given by

$$\Phi_{\nu(A)}^t(z) = e^{tA}z.$$

Let Λ_m be the hyperplane bundle on $\mathbf{P}(H_m)$, endowed with the Hermitian metric induced by b_m , and let g_m be the pull-back to M of the associated Fubini-Study metric on $\mathbf{P}(H_m)$. One can restrict $\nu(A)$ to M as a section of $\iota_m^*T\mathbf{P}(H_m)$, and then project orthogonally to $TM \subset \iota_m^*T\mathbf{P}(H_m)$ to get a smooth vector field $e_m(A)$ on M . This defines a map

$$e_m : V_m \rightarrow \Gamma(TM).$$

Recall that V_m has an inner product defined by (3). On the other hand, $\Gamma(TM)$ is equipped with the L^2 -inner product induced by the Kähler metric g_m :

$$(\eta, \xi)_m = \int_M g_m(\eta, \xi) \frac{\omega_m^n}{n!},$$

for all $\eta, \xi \in \Gamma(TM)$ (here ω_m is the Kähler form of g_m , i.e. the pull-back of the Fubini-study form to M). Thus one can form the adjoint operator

$$e_m^* : \Gamma(TM) \rightarrow V_m,$$

and finally define

$$\Delta_m = e_m^* \circ e_m. \quad (5)$$

Next lemma shows that the vector field $e_m(A)$ and the function $T_m^*(A)$ are related through the projectively induced Kähler metric g_m .

Lemma 4.1. *For all $A \in V_m$ one has*

$$e_m(A) = \text{grad}_m \frac{T_m^*(A)}{\rho_m},$$

where the gradient is taken with respect to the Riemannian metric g_m .

Proof. We have to show that $g_m(e_m(A), v) = v(T_m^*(A)/\rho_m)$ for all vector field $v \in \Gamma(TM)$. In order to do this, consider a smooth extension \tilde{v} of v to a smooth vector field of $\mathbf{P}(H_m)$. Since g_m is induced by the Fubini-Study metric g_{FS} on $\mathbf{P}(H_m)$, and $e_m(A)$ is the orthogonal projection of $\nu(A)$ on TM , one has

$$g_m(e_m(A), v) = \iota_m^* g_{FS}(\nu(A), \tilde{v}). \quad (6)$$

The right hand side of the equation above can be related to a function on $\mathbf{P}(H_m)$ naturally associated to A . Indeed we claim that $\nu(A)$ is the gradient of the function μ_A defined by

$$\mu_A(s) = \frac{b_m(As, s)}{b_m(s, s)}.$$

This is quite standard, but a proof of that fact is included at the end of the proof for convenience of the reader. Now we go ahead taking the claim for grant. From (6) one gets

$$g_m(e_m(A), v) = v(\iota_m^* \mu_A),$$

thus it remains to prove the identity

$$\iota_m^* \mu_A = T_m^*(A)/\rho_m. \quad (7)$$

To this end, let $\{s_\alpha\}$ be an orthonormal basis of H_m , so that the pull-back of μ_A to M is given by

$$\iota_m^* \mu_A(x) = \frac{\sum_{\alpha, \beta} s_\alpha(x) \overline{s_\beta(x)} b_m(As_\alpha, s_\beta)}{\sum_\gamma |s_\gamma(x)|^2},$$

where the ratio $\frac{s_\alpha(x) \overline{s_\beta(x)}}{\sum_\gamma |s_\gamma(x)|^2}$ is well defined and can be computed choosing an arbitrary Hermitian metric on the line bundle L^m . In particular, taking h^m it becomes $\frac{h^m(s_\alpha, s_\beta)(x)}{\sum_\gamma |s_\gamma|_{h^m}^2(x)}$, whence

$$\iota_m^* \mu_A(x) = \frac{\sum_{\alpha, \beta} \int_M h^m(As_\alpha, s_\beta)(z) h^m(s_\alpha, s_\beta)(x) \frac{\omega_z^n}{n!}}{\sum_\gamma |s_\gamma|_{h^m}^2(x)},$$

and the identity (7) follows by definition of ρ_m and Lemma 3.1.

Finally, in order to prove the claim above, let (z_α) be homogeneous coordinates on $\mathbf{P}(H_m)$ corresponding to the basis $\{s_\alpha\}$. The function μ_A then takes the form

$$\mu_A(z) = \frac{\bar{z} A z^t}{|z|^2},$$

where now $A = (A_{\alpha\beta})$ denotes the matrix that represents the endomorphism A with respect the chosen basis. The equality between $\nu(A)$ and the gradient of μ_A can be proved in local affine coordinates, but here we consider the projection of $H_m \setminus \{0\}$ on $\mathbf{P}(H_m)$, and the fact that $\nu(A)$, g_{FS} and μ^A lift to \mathbf{C}^* -invariant objects (which will be denoted with the same symbols). In particular one has

$$\nu(A) = \sum_{\alpha, \beta} A_{\alpha\beta} \left(z_\alpha \frac{\partial}{\partial z_\beta} + \bar{z}_\beta \frac{\partial}{\partial \bar{z}_\alpha} \right),$$

and

$$g_{FS} = \sum_i \frac{dz_i d\bar{z}_i}{|z|^2} - \sum_{i, j} \frac{\bar{z}_i z_j dz_i d\bar{z}_j}{|z|^4},$$

whence

$$i_{\nu(A)} g_{FS} = \sum_{\alpha, \beta} A_{\alpha\beta} \left(\frac{z_\alpha d\bar{z}_\beta + \bar{z}_\beta dz_\alpha}{|z|^2} - \frac{z_\alpha \bar{z}_\beta d|z|^2}{|z|^4} \right) = d\mu_A,$$

which proves the claim. \square

Next lemma characterizes the kernel of Δ_m .

Lemma 4.2. $\Delta_m(A) = 0$ if and only if A is a multiple of the identity.

Proof. By definition $\Delta_m = e_m^* \circ e_m$, and by Lemma 4.1 and its proof it follows $e_m(A) = \text{grad}_m \frac{T_m^*(A)}{\rho_m}$. Thus $\Delta_m(A) = 0$ if and only if $\frac{T_m^*(A)}{\rho_m} = c$ for some $c \in \mathbf{C}$. Let $I \in V_m$ be the identity. The identity $T_m^*(I) = \rho_m$ implies $\Delta_m(A) = 0$ if and only if $A - cI \in \ker T_m^*$, thus the thesis follows by injectivity of T_m^* [7, Proposition 3.4].

Alternatively, one can argue more geometrically as follows. In the proof of Lemma 4.1 has been introduced a smooth function μ_A on $\mathbf{P}(H_m)$ satisfying $\frac{T_m^*(A)}{\rho_m} = \iota_m^* \mu_A$. Thus by Lemma 4.1 one has $\Delta_m(A) = 0$ if and only if $\iota_m^* d\mu_A = 0$. Then the thesis follows by showing that the locus where $d\mu_A = 0$ contains no positive dimensional holomorphic submanifolds (or, in other words, $\ker d\mu_A$ is totally real), unless μ_A is constant. \square

Now we pass to give a more explicit description of the operator Δ_m . To this end fix an orthonormal basis $\{s_\alpha\}$ of H_m and let (z_i) be the corresponding homogeneous coordinates on $\mathbf{P}(H_m)$. Moreover this identifies V_m with the space of $\dim H_m \times \dim H_m$ complex matrices. Consider the map

$$\Psi_m : \mathbf{P}(H_m) \rightarrow V_m$$

defined by $\Psi_m(z) = \frac{z\bar{z}^t}{|z|^2}$. Note that by Lemma 3.1 follows that

$$\iota_m^* \text{tr}(A\Psi_m) = \frac{T_m^*(A)}{\rho_m} \quad (8)$$

for all $A \in V_m$. On the other hand, by definition of Δ_m one has

$$\text{tr}(\Delta_m(A)B^*) = \int_M g_m(e_m(A), e_m(B)) \frac{\omega_m^n}{n!},$$

thus by Lemma 4.1 together with (8) one gets

$$\int_M g_m(e_m(A), e_m(B)) \frac{\omega_m^n}{n!} = \int_M i\partial \text{tr}(\Psi_m A) \wedge \bar{\partial} \text{tr}(\Psi_m B^*) \wedge \frac{\omega_m^{n-1}}{(n-1)!}$$

whence

$$\text{tr}(\Delta_m(A)B^*) = \int_M i\partial \text{tr}(\Psi_m A) \wedge \bar{\partial} \text{tr}(\Psi_m B^*) \wedge \frac{\omega_m^{n-1}}{(n-1)!}.$$

Let

$$\Phi_m : \mathbf{P}(H_m) \rightarrow V_m^*$$

be the map obtained by composing Φ with the dual pairing induced by the Hermitian metric b_m on V_m . More explicitly one has

$$\Phi_m(z)(A) = \text{tr}(\Psi_m(z)A)$$

for all $A \in V_m$ and $z \in \mathbf{P}(H_m)$. By computation above we proved the following

Proposition 4.3. *Consider the $\text{End}(V_m)$ -valued differential form on $\mathbf{P}(H_m)$ defined by*

$$\Xi_m = i\partial\Phi_m \wedge \bar{\partial}\Psi_m \wedge e^{\omega_{FS}}.$$

Then it holds

$$\Delta_m = \int_M \Xi_m.$$

Here $e^{\omega_{FS}}$ is a mixed-degree form defined by the exponential series. Since $\omega_{FS}^k = 0$ for all $k \geq \dim H_m$, one has

$$e^{\omega_{FS}} = 1 + \omega_{FS} + \frac{\omega_{FS}^2}{2} + \cdots + \frac{\omega_{FS}^{\dim H_m - 1}}{(\dim H_m - 1)!}.$$

This implies that Ξ_m has mixed degree. More interestingly it depends just on the dimension of $\mathbf{P}(H_m)$ (and on choice of homogeneous coordinates) and it is independent of M .

Corollary 4.4.

$$\text{tr}(\Delta_m) = 2\pi n m^n \int_M \frac{\omega^n}{n!}.$$

Proof. Recall that we identified V_m with the space of $\dim H_m \times \dim H_m$ matrices by choosing an orthonormal basis $\{s_\alpha\}$ of H_m . The set of canonical matrices E_{ij} , then form an orthonormal basis of V_m . Thus by Proposition 4.3 one has

$$\begin{aligned} \text{tr}(\Delta_m) &= \sum_{\alpha,\beta} \langle \Delta_m(E_{\alpha\beta}), E_{\alpha\beta} \rangle \\ &= \sum_{\alpha,\beta} \int_M i\partial \left(\frac{z_\alpha \bar{z}_\beta}{|z|^2} \right) \wedge \bar{\partial} \left(\frac{z_\beta \bar{z}_\alpha}{|z|^2} \right) \wedge e^{\omega_{FS}} \\ &= \int_M \left(\frac{i\partial\bar{\partial}|z|^2}{|z|^2} - \frac{i\partial|z|^2 \wedge \bar{\partial}|z|^2}{|z|^4} \right) \wedge e^{\omega_{FS}} \\ &= 2\pi \int_M \omega_{FS} \wedge e^{\omega_{FS}} \\ &= 2\pi n \int_M \frac{\omega_m^n}{n!}, \end{aligned}$$

whence the thesis follows since ω_m is cohomologous to $m\omega$. \square

5 Proof of Theorem 1.1

First of all we recall some results on asymptotic expansions in Berezin-Toeplitz quantization.

Theorem 5.1. *There is a sequence $\{b_r\}$ of self-adjoint differential operators acting on $C^\infty(M)$ such that for any smooth function $f \in C^\infty(M)$ one has the asymptotic expansion*

$$T_m^* \circ T_m(f) = \sum_{r \geq 0} b_r(f) m^{n-r} + O(m^{-\infty}), \quad (9)$$

and for any $k, R \geq 0$ there exist constants $C_{k,R,f}$ such that

$$\left\| T_m^* \circ T_m(f) - \sum_{r=0}^R b_r(f) m^{n-r} \right\|_{C^k(M)} \leq C_{k,R,f} m^{n-R-1}.$$

Moreover one has

$$\begin{aligned} b_0(f) &= f, \\ b_1(f) &= \frac{\text{scal}(g)}{8\pi} f - \frac{1}{4\pi} \Delta f. \end{aligned}$$

Proof. See Ma and Marinescu [5]. The only fact one still needs to show is self-adjointness of operator b_r . It follows readily by self-adjointness of $T_m^* \circ T_m$ and expansion (9). Indeed one has

$$0 = \sum_{r=0}^R m^{n-r} \int_M \left(b_r(f) \bar{g} - f \overline{b_r(g)} \right) \frac{\omega^n}{n!} + O(m^{n-R-1}),$$

as $m \rightarrow +\infty$, for all $f, g \in C^\infty(M)$. □

Since the Bergman kernel satisfies $\rho_m = T_m^* \circ T^m(1)$, one recovers the well known asymptotic expansion [8, 5, 3, 6]

$$\rho_m = \sum_{r \geq 0} a_r m^{n-r} + O(m^{-\infty}), \quad (10)$$

where $a_r = b_r(1) \in C^\infty(M)$ depends polynomially in the curvature of g and its covariant derivatives. In particular

$$a_0 = 1, \quad a_1 = \frac{\text{scal}(g)}{8\pi}. \quad (11)$$

Lemma 5.2. *For any $A \in V_m$ one has*

$$\Delta_m(A) = T_m \left(\frac{\omega_m^n}{\rho_m \omega^n} \Delta_{g_m} \left(\frac{T_m^*(A)}{\rho_m} \right) \right),$$

where Δ_{g_m} denotes the Laplacian of metric g_m .

Proof. By definition of Δ_m , for any $B \in V_m$ it holds

$$\mathrm{tr}(\Delta_m(A)B^*) = \int_M g_m(e_m(A), e_m(B)) \frac{\omega_m^n}{n!},$$

whence, by Lemma 4.1 and integration by parts it follows

$$\mathrm{tr}(\Delta_m(A)B^*) = \int_M \Delta_{g_m} \left(\frac{T_m^*(A)}{\rho_m} \right) \frac{\overline{T_m^*(B)}}{\rho_m} \frac{\omega_m^n}{n!}.$$

The right hand side can be rewritten as

$$\int_M \frac{\omega_m^n}{\rho_m \omega^n} \Delta_{g_m} \left(\frac{T_m^*(A)}{\rho_m} \right) \overline{T_m^*(B)} \frac{\omega^n}{n!} = \mathrm{tr} \left(T_m \left(\frac{\omega_m^n}{\rho_m \omega^n} \Delta_{g_m} \left(\frac{T_m^*(A)}{\rho_m} \right) \right) B^* \right),$$

whence the statement follows by arbitrariness of B . \square

For any $f \in C^\infty(M)$, by Lemma above one has

$$T_m^* \circ \Delta_m \circ T_m(f) = T_m^* \circ T_m \left(\frac{\omega_m^n}{\rho_m \omega^n} \Delta_{g_m} \left(\frac{T_m^* \circ T_m(f)}{\rho_m} \right) \right), \quad (12)$$

thus the statement of Theorem 1.1 follows readily by Theorem 9 and asymptotic expansion (10). In particular one has

$$T_m^* \circ T_m(f) = m^n f + m^{n-1} b_1(f) + O(m^{n-2})$$

whence

$$\begin{aligned} \rho_m &= m^n + m^{n-1} a_1 + O(m^{n-2}), \\ \frac{T_m^* \circ T_m(f)}{\rho_m} &= f + m^{-1} (b_1(f) - a_1 f) + O(m^{-2}) \\ \omega_m &= m \omega + O(m^{-1}), \\ \frac{\omega_m^n}{\rho_m \omega^n} &= 1 - m^{-1} a_1 + O(m^{-2}), \\ \Delta_{g_m}(f) &= m^{-1} \Delta(f) + O(m^{-3}). \end{aligned}$$

Substituting in (12) finally gives

$$\begin{aligned} T_m^* \circ \Delta_m \circ T_m(f) &= T_m^* \circ T_m \left((1 - m^{-1} a_1) m^{-1} \Delta(f + m^{-1} (b_1(f) - a_1 f)) + O(m^{-3}) \right) \\ &= m^{-1} T_m^* \circ T_m \left(\Delta(f) + m^{-1} (\Delta b_1(f) - \Delta(a_1 f) - a_1 \Delta(f)) + O(m^{-2}) \right) \\ &= m^{-1} T_m^* \circ T_m \left(\Delta(f) + m^{-1} (\Delta b_1(f) - \Delta(a_1 f) - a_1 \Delta(f)) + O(m^{-2}) \right) \\ &= m^{n-1} \Delta f + m^{n-2} (\Delta(b_1(f)) + b_1(\Delta(f)) - \Delta(a_1 f) - a_1 \Delta(f)) + O(m^{n-3}). \end{aligned}$$

This obviously proves $P_0 = \Delta$ and, recalling that $b_1(f) = \frac{\mathrm{scal}(g)}{8\pi} f - \frac{1}{4\pi} \Delta(f)$

and $a_1 = \frac{\mathrm{scal}(g)}{8\pi}$, it gives

$$\begin{aligned} 8\pi P_1(f) &= \Delta(\mathrm{scal}(g)f - 2\Delta(f)) + \mathrm{scal}(g)\Delta(f) - 2\Delta^2(f) - \Delta(\mathrm{scal}(g)f) - \mathrm{scal}(g)\Delta(f) \\ &= -4\Delta^2(f), \end{aligned}$$

which concludes the proof.

6 Balanced metrics

Balanced metrics have been introduced by Donaldson in connection with the existence problem of constant scalar curvature Kähler metric on polarized manifolds [1]. Recall that a metric is called *m-balanced* if the density of state function ρ_m is constant. Note that the value of such a constant is not arbitrary for ρ_m satisfies $\int_M \rho_m \frac{\omega^n}{n!} = \dim H^0(M, L^m)$. Moreover, since in general one has $\omega_m = m\omega + \frac{i}{2\pi} \partial\bar{\partial} \log \rho_m$, ω is *m-balanced* if and only if $\omega_m = m\omega$. Thus, assuming that ω is *m-balanced*, by Lemma 5.2 for any $A \in V_m$ one has

$$\begin{aligned} \Delta_m(A) &= \rho_m^{-2} T_m \left(\frac{\omega_m^n}{\omega^n} \Delta_{g_m} (T_m^*(A)) \right) \\ &= \frac{m^{n-1} \left(\int_M \frac{\omega^n}{n!} \right)^2}{(\dim H^0(M, L^m))^2} T_m \circ \Delta \circ T_m^*(A), \end{aligned}$$

which proves Theorem 1.2.

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